

ON UNIVERSAL LEFT-STABILITY OF ϵ -ISOMETRIES

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ABSTRACT. Let X, Y be two real Banach spaces, and $\epsilon \geq 0$. A map $f : X \rightarrow Y$ is said to be a standard ϵ -isometry if $\|f(x) - f(y)\| - \|x - y\| \leq \epsilon$ for all $x, y \in X$ and with $f(0) = 0$. We say that a pair of Banach spaces (X, Y) is stable if there exists $\gamma > 0$ such that for every such ϵ and every standard ϵ -isometry $f : X \rightarrow Y$ there is a bounded linear operator $T : L(f) \equiv \overline{\text{span}}f(X) \rightarrow X$ such that $\|Tf(x) - x\| \leq \gamma\epsilon$ for all $x \in X$. $X(Y)$ is said to be left (right)-universally stable, if (X, Y) is always stable for every $Y(X)$. In this paper, we show that if a dual Banach space X is universally-left-stable, then it is isometric to a complemented w^* -closed subspace of $\ell_\infty(\Gamma)$ for some set Γ , hence, an injective space; and that a Banach space is universally-left-stable if and only if it is a cardinality injective space; and universally-left-stability spaces are invariant.

1. INTRODUCTION

In this paper, we study properties of universally-left-stable Banach spaces for ϵ -isometries, and search for characterizations of such stable spaces. We first recall definitions of isometry and ϵ -isometry.

Definition 1.1. Let X, Y be two Banach spaces, $\epsilon \geq 0$, and $f : X \rightarrow Y$ be a mapping.

(1) f is said to be an ϵ -isometry if

$$\|f(x) - f(y)\| - \|x - y\| \leq \epsilon \quad \text{for all } x, y \in X;$$

(2) In particular, if $\epsilon = 0$, then the 0-isometry f is simply called an isometry;

(3) We say that an (ϵ -) isometry f is standard if $f(0) = 0$.

Isometry and linear isometry. The study of properties of isometry between Banach spaces and its generalizations have continued for 80 years. The first celebrated result was due to Mazur and Ulam ([14], 1932): Every surjective isometry between two Banach spaces is necessarily affine. But the simple example: $f : \mathbb{R} \rightarrow \ell_\infty^2$ defined for t by $f(t) = (t, \sin t)$ shows that it is not true if an isometry is not surjective. For nonsurjective isometry, Figiel [7] showed the following remarkable theorem in 1968.

1991 *Mathematics Subject Classification.* Primary 46B04, 46B20, 47A58; Secondary 26E25, 46A20, 46A24.

Key words and phrases. ϵ -isometry, linear isometry, stability, injective space, Banach space.

[†] Support partially by NSFC, grant no.11071201 .

[‡] Support partially by NSFC, grant no.

Theorem 1.2 (Figiel). *Suppose f is a standard isometry from a Banach X to another Banach space Y . Then there is a linear operator $T : L(f) \rightarrow X$ with $\|T\| \leq 1$ such that $Tf(x) = x$, for all $x \in X$; or equivalently, $Tf = I_X$, the identity on X .*

In 2003, Godefroy and Kalton [9] studied the relationship between isometry and linear isometry, and showed the following deep theorem:

Theorem 1.3 (Godefroy-Kalton). *Suppose that X, Y are two Banach spaces.*

- (1) *If X is separable and there is an isometry $f : X \rightarrow Y$, then Y contains an isometric linear copy of X ;*
- (2) *If X is a nonseparable weakly compactly generated space, then there exist a Banach space Y and an isometry $f : X \rightarrow Y$, but X is not linearly isomorphic any subspace of Y .*

Recently, Cheng, Dai, Dong and Zhou [2] showed that for Banach spaces X and Y , if such an ε -isometry $f : X \rightarrow Y$ exists, then there is a linear isometry $U : X^{**} \rightarrow Y^{**}$; in particular, if Y is reflexive, then there is a linear isometry $U : X \rightarrow Y$.

ε -isometry and stability. In 1945, Hyers and Ulam proposed the following question[12] (see, also [16]): whether for every surjective ε -isometry $f : X \rightarrow Y$ with $f(0) = 0$ there exist a surjective linear isometry $U : X \rightarrow Y$ and $\gamma > 0$ such that

$$(1.1) \quad \|f(x) - Ux\| \leq \gamma\varepsilon, \text{ for all } x \in X.$$

After many years efforts of a number of mathematicians (see, for instance, [8], [10], [12], and [16]), the following sharp estimate was finally obtained by Omladič and Šemrl [16].

Theorem 1.4 (Omladič-Šemrl). *If $f : X \rightarrow Y$ is a surjective ε -isometry with $f(0) = 0$, then there is a surjective linear isometry $U : X \rightarrow Y$ such that*

$$\|f(x) - Ux\| \leq 2\varepsilon, \text{ for all } x \in X.$$

The study of nonsurjective ε -isometry has also brought to mathematicians' attention (see, for instance [5], [16], [17], [18] and [19]). Qian[17] first proposed the following problem in 1995.

Problem 1.5. *Whether there exists a constant $\gamma > 0$ depending only on X and Y with the following property: For each standard ε -isometry $f : X \rightarrow Y$ there is a bounded linear operator $T : L(f) \rightarrow X$ such that*

$$(1.2) \quad \|Tf(x) - x\| \leq \gamma\varepsilon, \text{ for all } x \in X.$$

Then he showed that the answer is affirmative if both X and Y are L_p spaces. Šemrl and Väisälä [18] further presented a sharp estimate of (1.2) with $\gamma = 2$ if both X and Y are L^p spaces for $1 < p < \infty$.

However, Qian (in the same paper [17]) presented the following simple counterexample.

Example 1.6 (Qian). *Given $\varepsilon > 0$, and let Y be a separable Banach space admitting a uncomplemented closed subspace X . Assume that g is a bijective mapping from X onto the closed unit ball B_Y of Y with $g(0) = 0$. We define a map $f : X \rightarrow Y$ by $f(x) = x + \varepsilon g(x)/2$ for all $x \in X$. Then f is an ε -isometry with $f(0) = 0$ and $L(f) = Y$. But there are no such T and γ satisfying (1.2).*

This disappointment makes us to search for (1) some weaker stability version and (2) some appropriate complementability assumption on some subspaces of Y associated with the mapping. Cheng, Dong and Zhang [3] showed the following theorems about the two questions.

Theorem 1.7 (Cheng-Dong-Zhang). *Let X and Y be Banach spaces, and let $f : X \rightarrow Y$ be a standard ε -isometry for some $\varepsilon \geq 0$. Then for every $x^* \in X^*$, there exists $\phi \in Y^*$ with $\|\phi\| = \|x^*\| \equiv r$ such that*

$$(1.3) \quad |\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon r, \text{ for all } x \in X.$$

For a standard ε -isometry $f : X \rightarrow Y$, let

$Y \supset E =$ the annihilator of all bounded linear functionals and bounded on

$$C(f) \equiv \overline{\text{co}}(f(X), -f(X)),$$

i.e.

$$(1.4) \quad E = \{y \in Y : \langle y^*, y \rangle = 0, y^* \in Y^* \text{ is bounded on } C(f)\}.$$

Theorem 1.8 (Cheng-Dong-Zhang). *Let X and Y be Banach spaces, and let $f : X \rightarrow Y$ be a standard ε -isometry for some $\varepsilon \geq 0$. Then*

(i) *If Y is reflexive and if E is α -complemented in Y , then there is a bounded linear operator $T : Y \rightarrow X$ with $\|T\| \leq \alpha$ such that*

$$\|Tf(x) - x\| \leq 4\varepsilon, \text{ for all } x \in X.$$

(ii) *If Y is reflexive, smooth and locally uniformly convex, and if E is α -complemented in Y , then there is a bounded linear operator $T : Y \rightarrow X$ with $\|T\| \leq \alpha$ such that the following sharp estimate holds*

$$\|Tf(x) - x\| \leq 2\varepsilon, \text{ for all } x \in X.$$

Universal stability spaces of ε -isometries. For study of the stability of ε -isometry of Banach spaces, the following two questions are very natural.

Problem 1.9. *Is there a characterization for the class of Banach spaces X satisfying that for every Banach space Y there is $\infty > \gamma > 0$ such that for every ε -isometry $f : X \rightarrow Y$ with $f(0) = 0$, there exists a bounded linear operator $T : L(f) \rightarrow X$ such that*

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \text{ for all } x \in X,$$

that is, inequality (1.2) holds. Every space X of this class is said to be a universal left-stability space.

Problem 1.10. *Can we characterize the class of Banach spaces Y , such that for every Banach space X there is $\infty > \gamma > 0$ such that for every ε -isometry $f : X \rightarrow Y$ with $f(0) = 0$, there exists a bounded linear operator $T : L(f) \rightarrow X$ such that*

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \quad \text{for all } x \in X.$$

Every space Y of this class is called a universal right-stability space.

Recently, Cheng, Dai, Dong and Zhou [2] studied properties of universal left-stability and right-stability spaces. As a result, they proved that up to linear isomorphism, universal right-stability spaces are just Hilbert spaces; Every injective space is universally left-stable, and a Banach space X which is linearly isomorphic to a subspace of ℓ_∞ is universally left-stable if and only if it is linearly isomorphic to ℓ_∞ . They also verified that a separable space X satisfies that the pair (X, Y) is stable for every separable Y if and only if X is linearly isomorphic to c_0 .

In this paper, we further show that if a dual Banach space X is universally left-stable, then it is isometric to a complemented w^* -closed subspace of $\ell_\infty(\Gamma)$ for some set Γ , hence, an injective space; and that a Banach space is universally left-stable if and only if it is a cardinality injective space. Therefore, the universal left-stability of Banach spaces is invariant under linear isomorphism.

All symbols and notations in this paper are standard. We use X to denote a real Banach space and X^* its dual. B_X and S_X denote the closed unit ball and the unit sphere of X , respectively. For a subspace $E \subset X$, E^\perp denotes the annihilator of E , i.e. $E^\perp = \{x^* \in X^* : \langle x^*, e \rangle = 0 \text{ for all } e \in E\}$. Given a bounded linear operator $T : X \rightarrow Y$, $T^* : Y^* \rightarrow X^*$ stands for its conjugate operator. For a subset $A \subset X$ (X^*), \overline{A} , $(w^*\text{-}\overline{A})$ and $\text{co}(A)$ stand for the closure (the w^* -closure), and the convex hull of A , respectively.

2. UNIVERSAL LEFT-STABILITY DUAL SPACES FOR ε -ISOMETRIES

In this section, we search for some properties of the class of universal left-stability spaces for ε -isometries. After discussion of cardinality of Banach spaces, we show that a dual Banach space is universally left-stable if and only if it is an injective space.

Recall that a Banach space $X(Y)$ is universally left(right)-stable if it satisfies that for every Banach space $Y(X)$ there is $\infty > \gamma > 0$ such that for every standard ε -isometry $f : X \rightarrow Y$, there exists a bounded linear operator $T : L(f) \rightarrow X$ such that

$$(2.1) \quad \|Tf(x) - x\| \leq \gamma\varepsilon, \quad \text{for all } x \in X.$$

As a result we show that (1) inequality (2.1) holds for every Banach space Y if and only if X is a cardinality injective Banach space; (2) if a dual Banach space X is universally left-stable, then it is isometric to a complemented w^* -closed subspace of $\ell_\infty(\Gamma)$ for some set Γ , hence, an injective space.

The following lemma is presented in [2].

Lemma 2.1. *Let X be a closed subspace of a Banach space Y . If $\text{card}(X) = \text{card}(Y)$, then for every $\varepsilon > 0$ there is a standard ε -isometry $f : X \rightarrow Y$ such that*

- (1) $L(f) \equiv \overline{\text{span}}f(X) = Y$;
- (2) X is complemented whenever f is stable.

A Banach space X is said to be injective if it has the following extension property: Every bounded linear operator from a closed subspace of a Banach space into X can be extended to be a bounded operator on the whole space. X is called isometrically injective if every such bounded operator has a norm-preserved extension (See, for example [1]). Goondner [11] introduced a family of Banach spaces coinciding with the family of injective spaces: for any $\lambda \geq 1$, a Banach space X is a P_λ -space if, whenever X is isometrically embedded in another Banach space, there is a projection onto the image of X with norm not larger than λ . The following result was due to Day [4] (see, also, Wolfe [20], Fabian et al. [6], p. 242).

Proposition 2.2. *A Banach space X is (isometrically) injective if and only if it is a P_λ (P_1)-space for some $\lambda \geq 1$.*

Goondner [11], Nachbin [15] and Kelley [13] characterized the isometrically injective spaces.

Theorem 2.3 (Goodner-Kelley-Nachbin, 1949-1952). *A Banach space is isometrically injective if and only if it is isometrically isomorphic to the space of continuous functions $C(K)$ on an extremely disconnected compact Hausdorff space K , i.e. the space K such that the closure of any open set is open in K .*

Remark 2.4. For any set Γ endowed with the discrete metric topology, $\ell_\infty(\Gamma)$ is isomorphic to $C(K_\Gamma)$, where K_Γ is the Stone-Čech compactification of Γ . By Goodner-Kelley-Nachbin's theorem, $\ell_\infty(\Gamma)$ is an isometrically injective space.

Lemma 2.5. *Suppose that X is a Banach space. Then*

$$(2.2) \quad \text{dens}(X) \geq w^*\text{-dens}(B_{X^*}) \geq w^*\text{-dens}(X^*).$$

Proof. It is trivial if $\dim X < \infty$. Assume that $\dim X = \infty$. Note $\text{dens}X = \text{dens}S_X$. Let $(x_\alpha) \subset S_X$ be a dense subset with $\text{card}(x_\alpha) = \text{des}S_X$, and let ϕ be a selection of the subdifferential mapping $\partial\|\cdot\| : X \rightarrow 2^{S_{X^*}}$ of the norm $\|\cdot\|$ defined by

$$\partial\|x\| = \{x^* \in X^* : \|x + y\| - \|x\| \geq \langle x^*, y \rangle, \text{ for all } y \in X\}.$$

Then

$$(2.3) \quad w^*\text{-}\overline{\text{co}}(\phi(x_\alpha)) = B_{X^*}.$$

Since $\phi(x_\alpha)$ is a norming set of X , i.e.

$$\|x\| = \sup_\alpha \langle \phi(x_\alpha), x \rangle, \text{ for all } x \in X,$$

and since $\dim X = \infty$, $\text{card}(\phi(x_\alpha)) = \infty$. Hence,

$$\begin{aligned}\text{dens}(X) &= \text{dens}(S_X) = \text{card}(x_\alpha) \geq \text{card}(\phi(x_\alpha)) = \text{dens}\{\text{co}(\phi(x_\alpha))\} \\ &\geq w^*\text{-dens}\{w^*\text{-}\overline{\text{co}}(\phi(x_\alpha))\} = w^*\text{-dens}(B_{X^*}) \geq w^*\text{-dens}(X^*).\end{aligned}$$

Equality (2.3) is used to the last equality above. \square

We should mention here that the inequalities in Lemma 2.5 can be proper. For example, let $X = \ell_\infty$. Then $\text{dens}(X) = \text{card}[0, 1]$, but B_{X^*} is w^* -separable. On the other hand, we put an equivalent norm $\|\cdot\|$ on ℓ_∞ by

$$\|\|x\|\| = \frac{1}{2}(\|x\|_\infty + \limsup_n |x(n)|), \text{ for all } x = (x(n)) \in \ell_\infty.$$

Then $X^* = \ell_1 \oplus c_0^\perp$ is w^* -separable, but its closed unit ball B_{X^*} is not w^* -separable.

Lemma 2.6. *Let X be a Banach space with $\dim X \geq 1$, and let $\Omega = \text{dens}(X)$. Then*

$$\text{card}(X) = \text{card}(c_0(\Omega)).$$

Proof. If X is separable, then $\text{dens}(X) = \aleph_0$ and

$$\text{card}(X) = \Omega^{\aleph_0} = \aleph_0^{\aleph_0} = \aleph^{\aleph_0} = \text{card}(c_0(\aleph_0)) = \text{card}(c_0(\Omega)).$$

If X is not separable, then

$$\aleph \leq \Omega = \Omega^{\aleph_0} = \text{card}(X).$$

Let e_ω (for all $\omega \in \Omega$) be the standard unit vectors in $c_0(\Omega)$.

$$\begin{aligned}\text{card}(c_0(\Omega)) &= \text{dens}(c_0(\Omega))^{\aleph_0} = \text{dens}(c_0(\Omega)) \\ &= \text{dens}(\text{span}(e_\omega)_{\omega \in \Omega}) = \sum_{n=1}^{\infty} (\aleph_0 \cdot \Omega)^n = \Omega.\end{aligned}$$

\square

Theorem 2.7. *Suppose that X is a universal left-stability spaces. Then there is an injective conjugate space V such that $X \subset V \subset X^{**}$.*

Proof. Let $(x_\gamma^*) \subset B_{X^*}$ be a w^* -dense subset of B_{X^*} with $\text{card}(x_\gamma^*) = w^*\text{-dens} B_{X^*} \equiv \Gamma$, where Γ denotes the cardinality of the w^* -density of B_{X^*} . By Lemma 2.5, $\text{dens}(X) \geq \Gamma$. Let $\Phi : X \rightarrow \ell_\infty(\Gamma)$ be defined by

$$\Phi(x) = (\langle x_\gamma^*, x \rangle)_{\gamma \in \Gamma}, \quad x \in X.$$

Clearly, Φ is a linear isometry. Now, there are two w^* -topologies on X : One is the w^* -topology of X^{**} restricted to $X \subset X^{**}$, which we denote by τ_{w^*} ; the other is, acting as a subspace of $\ell_\infty(\Gamma)$, the w^* -topology of $\ell_\infty(\Gamma) = \ell_1(\Gamma)^*$ restricted to $\Phi(X)$, which we denote by $\tau_{w^*, \infty}$. Let $V = \tau_{w^*, \infty}\text{-}\overline{\Phi(X)}$, the $\tau_{w^*, \infty}$ -closure of $\Phi(X)$ in $\ell_\infty(\Gamma)$. Without loss of generality, we assume $\Phi(X) = X$. Note $X^{**} =$ the w^* -closure of X in $\ell_\infty(\Gamma)^{**} = \ell_\infty \bigoplus (c_0(\Gamma)^\perp)^*$. It is easy to get $X \subset V \subset X^{**}$.

Let $Z = \overline{\text{span}}(X \cup c_0(\Gamma))$. By Lemma 2.5 and Lemma 2.6, $\text{card}(X) = \text{card}(Z)$.

Let $g : X \rightarrow B_Z$ be a bijection with $g(0) = 0$, and let $f : X \rightarrow Z$ be defined by

$$f(x) = x + \varepsilon/2 \cdot g(x), \quad \text{for all } x \in X.$$

Clearly, f is a standard ε -isometry. Note $\lim_{n \rightarrow \infty} f(nx)/n = x$, for all $x \in X$. We see that $L(f) = Z$. By universal left-stability assumption, there exist a bounded linear operator $T : Z \rightarrow X$ and a positive number $\gamma > 0$ such that

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \quad \text{for all } x \in X.$$

It is easy to see that T is surjective. Thus its biconjugate operator $T^{**} : Z^{**} \rightarrow X^{**}$ is surjective and w^* -to- w^* continuous with $\|T^{**}\| = \|T\|$ and with $T^{**}|_Z = T$. Since Z isometrically contains $c_0(\Gamma)$, Z^{**} (isometrically) contains $\ell_\infty(\Gamma)$. Note that Z acting as a subspace of $\ell_\infty(\Gamma)$, its w^* -closure in $\ell_\infty(\Gamma)^{**} = \ell_\infty(\Gamma) \oplus (c_0(\Gamma)^\perp)^*$ is just Z^{**} . Let $P : \ell_\infty(\Gamma)^{**} \rightarrow \ell_\infty(\Gamma)$ be the natural projection. Clearly, P is w^* -to- w^* continuous with $\|P\| = 1$. On the other hand, since Z contains $c_0(\Gamma)$, it is a w^* -dense subspace of $\ell_\infty(\Gamma)$. Therefore, its w^* -closure in $\ell_\infty(\Gamma) = \ell_1(\Gamma)^*$ is just the whole space $\ell_\infty(\Gamma)$. We claim that $PT^{**}\ell_\infty(\Gamma) = V$.

Let $S = PT^{**}$. Then $S : Z^{**} \rightarrow \ell_\infty(\Gamma)$ is w^* -to- w^* continuous. It suffices to show $S(\ell_\infty(\Gamma)) \subset V$. Given $z \in \ell_\infty(\Gamma)$, since $B_Z(B_{c_0(\Gamma)}$, resp.) is w^* -dense in $B_{Z^{**}}(B_{\ell_\infty(\Gamma)}$, resp.), there is a net $(z_\alpha) \subset \|z\|B_Z$ such that $z_\alpha \rightarrow z$ in both the w^* topologies of $\ell_\infty(\Gamma)$ and $\ell_\infty(\Gamma)^{**}$. Thus, $S(z_\alpha) \rightarrow S(z) \in \ell_\infty(\Gamma)$, in the w^* -topology of $\ell_\infty(\Gamma)$. Since $S(z_\alpha) = PT^{**}(z_\alpha) = PT(z_\alpha) = T(z_\alpha) \in X \subset V$, and since V is a w^* -closed subspace of $\ell_\infty(\Gamma)$, we see $S(z) \in V$.

We have proven that V is a complemented w^* -closed subspace of $\ell_\infty(\Gamma)$, which in turn entails that V is a conjugate injective space. \square

Corollary 2.8. *Suppose that X is a universally left-stable conjugate Banach spaces. Then it is an injective space.*

Proof. Suppose that W is a Banach space with $W^* = X$. Let Γ be the cardinality of the w^* -density of $B_{X^*} = B_{W^{**}}$ and let $\Omega = \text{dens}(W) = \text{dens}(B_W)$. Then, by the separation theorem of convex sets, $\Omega = \Gamma$. Indeed, on one hand, any dense subset of B_W is necessarily w^* -dense in $B_{W^{**}}$; on the other hand, for any subset S of B_W with cardinality less than Ω we see that $U \equiv \overline{\text{span}}S$ is a closed proper subspace of W . Therefore, by the separation theorem, there exists non-zero functional $\phi \in W^* = X$ such that $\phi(U) = \{0\}$. Hence, U is not w^* -dense in $W^{**} = X^*$, which entails that $S \subset B_W$ is not w^* -dense in B_{X^*} . Assume that $(w_\gamma)_{\gamma \in \Gamma} \subset B_W$ is a dense subset of B_W . Let $\Phi : X \rightarrow \ell_\infty(\Gamma)$ be defined by

$$\Phi(x) = (\langle w_\gamma, x \rangle)_{\gamma \in \Gamma}.$$

Clearly, Φ is a w^* -to- w^* continuous linear isometry. Conversely, since (w_γ) is dense in B_W , Φ^{-1} is also w^* -to- w^* continuous. Thus, $\Phi(X)$ is a w^* -closed space of $\ell_\infty(\Gamma)$. According to definition of the space V in Theorem 2.7, $\Phi(X) = V$. So that $\Phi(X)$ is an injective space, hence, X is also injective. \square

3. A CHARACTERIZATION OF UNIVERSAL-LEFT-STABILITY SPACES

Definition 3.1. A Banach space X is said to be cardinality injective, if there exists a constant $\lambda \geq 0$ such that for every Banach space Y isometrically containing X and with the same cardinality as X , i.e., $\text{card}(Z) = \text{card}(X)$, there is a projection $P : Y \rightarrow X$ with $\|P\| \leq \lambda$.

Using the same procedure of Day [4], we can show that X is cardinality injective if and only if it has the following extension property: Every bounded linear operator from a closed subspace of a Banach space Y with $\text{card}(Y) \leq \text{card}(X)$ into X can be extended to be a bounded operator on the whole space. Thus, we have the following property.

Proposition 3.2. *A Banach space isomorphic to a cardinality injective space is again a cardinality injective space.*

The following theorem says that a Banach space is universally left-stable if and only if it is cardinally injective.

Theorem 3.3. *Let X be a Banach space. Then a sufficient and necessary condition for that there is $\gamma > 0$ such that for every Banach space Y , every nonnegative number ε and for every standard ε -isometry $f : X \rightarrow Y$ there exists a bounded linear operator $T : L(f) \rightarrow X$ satisfying*

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \quad \text{for all } x \in X$$

is that X is a cardinality injective space.

Proof. Sufficiency. Assume that X is a cardinality injective Banach space. Then there exists $\alpha > 0$ such that for every Banach space Z isometrically containing X there is a projection $P : Z \rightarrow X$ such that $\|P\| \leq \alpha$. We can assume that X is a closed subspace of $\ell_\infty(\Gamma)$ for some set Γ ; otherwise, we can identify X for $J_X(X)$ as a closed subspace of $\ell_\infty(\Gamma)$, where Γ denotes the closed ball B_{X^*} of X^* . Given any $\beta \in \Gamma$, let $\delta_\beta \in \ell_\infty(\Gamma)^*$ be defined for $x = (x(\gamma))_{\gamma \in \Gamma} \in \ell_\infty(\Gamma)$ by $\delta_\beta(x) = x(\beta)$.

Assume that $f : X \rightarrow Y$ be an ε -isometry with $f(0) = 0$. For every $x^* \in X^*$, by Theorem 1.7, there is $\phi \in Y^*$ with $\|\phi\| = \|x^*\|$ such that

$$(3.2) \quad |\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon\|x^*\|, \quad \text{for all } x \in X.$$

In particular, letting $x^* = \delta_\gamma$ in (3.2) for every fixed $\gamma \in \Gamma$, we obtain a linear functional $\phi_\gamma \in Y^*$ satisfying (3.2) with $\|\phi_\gamma\| = \|\delta_\gamma\|_X \leq 1$. Therefore,

$$S(y) = (\phi_\gamma(y))_{\gamma \in \Gamma}, \quad \text{for every } y \in Y$$

defines a linear operator $S : Y \rightarrow \ell_\infty(\Gamma)$ with $\|S\| \leq 1$. Let $Z = \text{span}[S(f(X)) \cup X]$. Then $\overline{Z} \supset X$ and $\text{card}(Z) = \text{card}(X)$. Since X is a cardinality injective space, there is a projection $P : \overline{Z} \rightarrow X$ with $\|P\| \leq \alpha < \infty$.

Let $T(y) = P(S(y))$, for all $y \in L(f)$, and note $P|_X = I_X$, the identity from X to itself. Then $\|T\| \leq \|P\|\|S\| \leq \alpha$ and for all $x \in X$,

$$\|Tf(x) - x\| = \|P(\phi_\gamma(f(x)))_{\gamma \in \Gamma} - (\delta_\gamma(x))_{\gamma \in \Gamma}\|$$

$$\begin{aligned}
&= \|P(\phi_\gamma(f(x)))_{\gamma \in \Gamma} - P((\delta_\gamma(x))_{\gamma \in \Gamma})\| \\
&\leq \|P\| \cdot \|(\phi_\gamma(f(x)))_{\gamma \in \Gamma} - (\delta_\gamma(x))_{\gamma \in \Gamma}\|_\infty \leq 4\alpha\varepsilon.
\end{aligned}$$

We finish the proof of the sufficiency by taking $\gamma = 4\alpha$.

Necessity. Suppose, to the contrary, that X is not a cardinality injective space. Then, there is a Banach space Y containing X and a dense subspace Z of Y such that $\text{card}(Z) = \text{card}(X)$, but X is not complemented in Y . By Lemma 2.1, there is a standard ε -isometry $f : X \rightarrow Y$ which is not stable. \square

Corollary 3.4. *The universally-left-stability of Banach spaces is invariant under linear isomorphism.*

Proof. It suffices to note Theorem 3.3 and Proposition 3.2. \square

Corollary 3.5. *Let X be a Banach space. If for every Banach space Y , every nonnegative number ε and for every standard ε -isometry $f : X \rightarrow Y$ there exist $\infty > \gamma \geq 0$ and a bounded linear operator $T : L(f) \rightarrow X$ satisfying*

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \quad \text{for all } x \in X,$$

then X is universally left-stable, i.e. the positive number γ can be chosen depending only on X .

Proof. By the Theorem 3.3, it suffices to show that X is a λ -cardinally injective. But it is the same proof of the necessity of the theorem above. \square

Remark 3.6. Cheng, Dai, Dong and Zhou [2] recently showed that every injective Banach space is universally left-stable. By this result and Corollary 2.8, we get that a dual Banach space is injective if and only if it is universally left-stable. This and Theorem 3.3 entail that a dual Banach space is cardinality injective if and only if it is injective. But we do not know whether it is true in general.

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